

EXISTENCE AND SYMMETRY RESULTS FOR COMPETING VARIATIONAL SYSTEMS

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ABSTRACT. In this paper we consider a class of gradient systems of type

$$-c_i \Delta u_i + V_i(x)u_i = P_{u_i}(u), \quad u_1, \dots, u_k > 0 \text{ in } \Omega, \quad u_1 = \dots = u_k = 0 \text{ on } \partial\Omega,$$

in a bounded domain $\Omega \subseteq \mathbb{R}^N$. Under suitable assumptions on V_i and P , we prove the existence of ground-state solutions for this problem. Moreover, for $k = 2$, assuming that the domain Ω and the potentials V_i are radially symmetric, we prove that the ground state solutions are foliated Schwarz symmetric with respect to antipodal points. We provide several examples for our abstract framework.

Keywords. *Competitive systems, elliptic gradient systems, foliated schwarz symmetry, ground states solutions, positive solutions.*

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 1$ and $P : \mathbb{R}^k \rightarrow \mathbb{R}$ a \mathcal{C}^2 -function for some positive integer k . Moreover, let $V_i \in L^\infty(\Omega)$ for $i = 1, \dots, k$, and let c_1, \dots, c_k denote positive constants. In this paper, we will be concerned with the Dirichlet problem

$$\begin{cases} -c_i \Delta u_i + V_i(x)u_i = P_{u_i}(u), & u_1, \dots, u_k > 0 & \text{in } \Omega, \\ u_1 = u_2 = \dots = u_k = 0 & & \text{on } \partial\Omega. \end{cases} \quad (1)$$

where P_{u_i} stands for $\frac{\partial P}{\partial u_i}$. Note that the elliptic system in (1) is of gradient type. Under suitable assumptions on P , we will prove the existence of ground state solutions of (1) which can be found by minimizing an associated functional over a natural constraint. In certain cases, we will also provide a minimax characterization of the solutions, and in the case $k = 2$ we will deduce symmetry properties of the solutions from this characterization in the case where Ω is a radially symmetric domain in \mathbb{R}^N and the potentials V_i are also radially symmetric. We point out that we are only interested in nontrivial solutions of (1) in the sense that $u_i \not\equiv 0$ for $i = 1, \dots, k$. Consider the Hilbert space $\mathcal{H} := H_0^1(\Omega; \mathbb{R}^k)$ and the *Nehari type set*

$$\mathcal{N} := \left\{ u \in \mathcal{H} : u_i \geq 0, u_i \not\equiv 0 \text{ and } \int_{\Omega} (c_i |\nabla u_i|^2 + V_i(x)u_i^2) dx = \int_{\Omega} P_{u_i}(u)u_i dx \text{ for } i = 1, \dots, k. \right\} \quad (2)$$

If Ω is of class \mathcal{C}^1 and $u \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^k)$ is a classical solution of (1) with nontrivial components, then we may multiply the i -th equation in (1) with u_i and integrate by parts to see that u belongs to \mathcal{N} . Therefore \mathcal{N} is a natural constraint for solutions of (1). If moreover P satisfies suitable growth assumptions (see assumption (P1) below), then weak solutions are precisely

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the critical points of the energy functional $E : \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$E(u) = \frac{1}{2} \sum_{i=1}^k \int_{\Omega} (c_i |\nabla u_i|^2 + V_i(x) u_i^2) dx - \int_{\Omega} P(u(x)) dx,$$

A natural but not straightforward approach to find solutions of (1) is to minimize E on \mathcal{N} . This approach has been carried out successfully in the scalar case $k = 1$ (see e.g. the recent survey [21] and the references therein) and also for special classes of elliptic systems, see e.g. [7, 9, 12, 13]. In this paper we will consider a general class of functions P for which we can show that minimizers of E on \mathcal{N} exist and are indeed solutions of (1). Such solutions then also minimize the energy E among the set of solutions and therefore will be called *ground state solutions*. It is natural to expect that in the case where the underlying domain Ω and the potentials V_i are radially symmetric, these ground state solutions inherit at least partially the symmetry of Ω and V_i . In a general framework, a principle of symmetry inheritance of constrained minimizers of integral functionals was recently proved by Mariş [14]. In particular, the following statement can be deduced from [14, Theorem 1]:

If $k \leq N - 2$, Ω is radially symmetric and every minimizer of E on \mathcal{N} is a solution of (1) (and therefore a C^1 -function on Ω), then every minimizer of E on \mathcal{N} is radially symmetric with respect to a k -dimensional subspace W of \mathbb{R}^N , i.e., $u(x) = u(y)$ for every $x, y \in \Omega$ such that $x - y \in W^\perp$ and $\text{dist}(x, W) = \text{dist}(y, W)$.

We stress that this symmetry result does not depend on further assumptions on P . Much more is known in the special case where the underlying domain Ω is a ball, $V_i \equiv 0$, and the system (1) is *cooperative*, i.e. $P_{u_i u_j} = \frac{\partial^2 P}{\partial u_i \partial u_j} \geq 0$ for all $u \in \mathbb{R}^k$ and $i, j = 1, \dots, k$. In this case, every solution of (1) is in fact radially symmetric (with respect to $W = \{0\}$) and decreasing in the radial variable by the general symmetry result of Troy [23] for cooperative systems. We note that Troy's result is proved via the moving plane method and therefore relies strongly on the cooperativity assumption. In the present paper, we are interested in the complementary case of non-cooperative competition-type systems which have been at the center of growing attention in recent years, see e.g. [3, 6–9, 15, 16, 22] and references therein. In this case ground state solutions are nonradial in general even if the underlying data is radially symmetric (see Remark 5.4 below). Nevertheless, we shall see below that, at least in the two-component case, the competitive character of the system also leads to an improvement of Mariş' symmetry result mentioned above.

In order to state our main results, we define the cone

$$C^+ := \{u = (u_1, \dots, u_k) \in \mathbb{R}^k : u_i \geq 0 \text{ for all } i\}$$

and we impose the following assumptions on the functions V_i , $i = 1, \dots, k$ and P :

- (P0) $V_i \in L^\infty(\Omega)$ and $\inf_{\Omega} V_i > -c_i \lambda_1(\Omega)$ for $i = 1, \dots, k$, where λ_1 is the first Dirichlet eigenvalue of the Laplacian on Ω ;
- (P1) there exists $2 < p < 2^*$ such that

$$|P_{u_i u_j}(u)| \leq C(1 + \sum_{i=1}^k |u_i|^{p-2}) \quad \text{for every } u \in C^+, i, j = 1, \dots, k,$$

where $2^* = 2N/(N - 2)$ if $N \geq 3$, $2^* = +\infty$ otherwise.

- (P2) $P(0, \dots, 0) = 0$ and $P_{u_i}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_k) = 0$ for every $u \in C^+$ and $i = 1, \dots, k$.

(P3) $P_{u_i}(u)u_i \leq P_{u_i}(0, \dots, u_i, \dots, 0)u_i \neq 0$ for every $u \in C^+$ and every $i \in \{1, \dots, k\}$ with $u_i \neq 0$;

(P4) there exists $\alpha > 0$ such that the matrix

$$M(u) := \left(\delta_{ij}(1 + \alpha)P_{u_i}(u)u_i - P_{u_i u_j}(u)u_i u_j \right)_{i,j=1,\dots,k}$$

$$= \begin{pmatrix} (1 + \alpha)P_{u_1}(u)u_1 - P_{u_1 u_1}(u)u_1^2 & \dots & -P_{u_1 u_k}(u)u_1 u_k \\ -P_{u_1 u_2}(u)u_1 u_2 & \dots & -P_{u_2 u_k}(u)u_2 u_k \\ \vdots & \ddots & \vdots \\ -P_{u_1 u_k}(u)u_1 u_k & \dots & (1 + \alpha)P_{u_k}(u)u_k - P_{u_k u_k}(u)u_k^2 \end{pmatrix}$$

is negative semidefinite for $u \in C^+$.

Note that condition (P3) is a weak competitiveness assumption for the system (1). Condition (P4) can be seen has a generalization of an Ambrosetti-Prodi condition, and it has appeared before in the papers [7, 8] (actually, we will see in the last section that our assumptions are more general than the ones considered in the mentioned papers). We now put

$$c = \inf_{u \in \mathcal{N}} E(u), \quad (3)$$

where \mathcal{N} was defined in (2). Our first main result shows that minimizers of E on \mathcal{N} exist and are ground state solutions of (1). More precisely, we have:

Theorem 1.1. *Suppose that (P0)–(P4) holds. Then there exists $u = (u_1, \dots, u_k) \in \mathcal{H}$, with $u_i > 0$ for all i , such that*

$$E'(u) = 0 \quad \text{and} \quad E(u) = c.$$

Moreover, every minimizer of $E|_{\mathcal{N}}$ is a solution of (1).

It is worth discussing the scalar case $k = 1$ in some detail. In this case, the assumptions (P1)–(P4) above reduce to requiring $P \in C^2(\mathbb{R})$, $P(0) = P'(0) = 0$ as well as $|P''(u)| \leq C(1 + |u|^{p-2})$ and $0 < (1 + \alpha)P'(u) \leq P''(u)u$ for every $u > 0$ with p as in (P1) and some $\alpha > 0$. From these assumptions, it follows that 0 is a local minimum for the corresponding functional $E : H_0^1(\Omega) \rightarrow \mathbb{R}$. Moreover, for any $u \in H_0^1(\Omega) \setminus \{0\}$ the function $\varphi_u : [0, \infty) \rightarrow \mathbb{R}$, $\varphi_u(t) = E(tu)$ satisfies $\varphi_u(0) = 0$ and $\varphi_u(t) \rightarrow -\infty$ as $t \rightarrow \infty$, and φ_u has a unique maximum t_u such that $t_u u \in \mathcal{N}$. As a consequence, in the scalar case $k = 1$ we have the minimax characterization

$$c = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \sup_{t \geq 0} E(tu) \quad (4)$$

In the case $k > 1$ assumptions (P1)–(P4) do not impose such a simple mountain pass geometry for the functional E . Nevertheless, for a large class of functions P satisfying these assumptions one may generalize the minimax characterization (4). For this we consider the set

$$\mathcal{M} = \{u \in \mathcal{H} : u_i \geq 0, u_i \neq 0 \text{ for } i = 1, \dots, k \text{ and } E(t_1 u_1, \dots, t_k u_k) \rightarrow -\infty \text{ as } |t_1| + \dots + |t_k| \rightarrow +\infty\}.$$

We then have:

Theorem 1.2. *Suppose that (P0)–(P4) and the following condition holds.*

(P5) $P_{u_i u_i}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_k) \leq 0$ for $u \in C^+$ and $i = 1, \dots, k$.

Then we have

$$c \leq \inf_{u \in \mathcal{M}} \sup_{t_1, \dots, t_k \geq 0} E(t_1 u_1, \dots, t_k u_k). \quad (5)$$

If moreover there exists $u \in \mathcal{N}$ with $E(u) = c$ and such that $u \in \mathcal{M}$, then equality holds in (5).

The crucial step in the proof of Theorem 1.2 is to prove that, for fixed $u \in \mathcal{M}$, the function

$$(t_1, \dots, t_k) \mapsto E(t_1 u_1, \dots, t_k u_k)$$

has precisely one critical point in $C^+ \setminus \{0\}$ which is a global maximum of this function in $C^+ \setminus \{0\}$. This fact will also be used in the proof of our main symmetry result Theorem 1.3 below. While assumptions (P1)–(P5) do not guarantee that every function $u \in \mathcal{H}$ with $u_i \not\equiv 0$ for $i = 1, \dots, k$ is contained in \mathcal{M} , below we will present classes of functions P ensuring that $\mathcal{N} \subset \mathcal{M}$, so that equality holds in (5). One explicit example we consider is the class of functions

$$P \in \mathcal{C}^2(\mathbb{R}^k), \quad P(u_1, \dots, u_k) = \sum_{i=1}^k \frac{\lambda_i}{p} |u_i|^p - \sum_{\substack{i,j=1 \\ i \neq j}}^n \beta_{ij} |u_i|^{q_i} |u_j|^{q_j} \quad (6)$$

which leads to the system

$$\begin{cases} -c_i \Delta u_i + V_i(x) u_i = \lambda_i u_i^{p-1} - q_i u_i^{q_i-1} \sum_{j \neq i} \beta_{ij} u_j^{q_j}, \\ u_i \in H_0^1(\Omega), \quad u_i > 0 \text{ in } \Omega, \end{cases} \quad (7)$$

Here we assume $2 < p < 2^*$ and

$$\lambda_i > 0, \quad \beta_{ij} = \beta_{ji} \geq 0, \quad q_i \geq 2 \quad \text{and} \quad p \geq q_i + q_j \quad \text{for } i, j = 1, \dots, k, \quad j \neq i. \quad (8)$$

A system of this kind also appears in [18]. We point out that one may divide out (or replace by arbitrary positive constants) the factors q_i in front of the sums in (7) without changing the nature of the system simply by adjusting the values of c_i , $V_i(x)$ and λ_i . Therefore, the cubic system

$$-\Delta u_i + V_i(x) u_i = \lambda_i u_i^3 - u_i \sum_{j \neq i} \beta_{ij} u_j^2, \quad i = 1, \dots, k,$$

arising in the theory of Bose-Einstein condensation and in nonlinear optics (see e.g. [17, 19]) can be seen as a special case of (7).

In Section 5 below we will show that the class of functions P given by (6) satisfies (P1)–(P5), whereas we also have $\mathcal{N} \subset \mathcal{M}$ so that equality holds in (5).

Our final main result is concerned with symmetry properties of ground state solutions of (1) in the case where the underlying domain $\Omega \subset \mathbb{R}^N$ and the potentials V_i are radial. For this we recall the notion of foliated Schwarz symmetry. A function $u : \Omega \rightarrow \mathbb{R}$ is called foliated Schwarz symmetric with respect to some unit vector $p \in \mathbb{R}^N$ if for a.e. $r > 0$ such that $\partial B_r(0) \subset \Omega$ and for every $c \in \mathbb{R}$ the restricted superlevel set $\{x \in \partial B_r(0) : u(x) \geq c\}$ is either equal to $\partial B_r(0)$ or to a geodesic ball in $\partial B_r(0)$ centered at rp . In other words, u is foliated Schwarz symmetric if u is axially symmetric with respect to the axis $\mathbb{R}p$ (i.e. radially symmetric with respect to the subspace spanned by p in the sense defined above) and nonincreasing in the polar angle $\theta = \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$. We have to restrict our attention to the case of two components, and we will write (u, v) in place of (u_1, u_2) in the following. Hence we consider the system

$$\begin{cases} -c_1 \Delta u + V_1(x) u = P_u(u, v), & -c_2 \Delta v + V_2(x) v = P_v(u, v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, & u = v = 0 & \text{on } \partial \Omega. \end{cases} \quad (9)$$

Theorem 1.3. *Suppose that $\Omega \subset \mathbb{R}^N$ is a radial domain, that V_1, V_2 are radial functions, and suppose that (P0)–(P5) hold for $k = 2$. Suppose moreover that*

(P6) $P_{uv}(s, t) < 0$ for every $s, t > 0$.

Let $(u, v) \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2)$ be a classical solution of (9) minimizing $E|_{\mathcal{N}}$. If $(u, v) \in \mathcal{M}$, then u and v are foliated Schwarz symmetric with respect to antipodal points.

We note that – for a class of systems with competition – Theorem 1.3 improves the symmetry result of Mariş [14] which in contrast to Theorem 1.3 only yields radial symmetry with respect to a two-dimensional subspace. For a larger number $k \geq 3$ of components, it remains open whether the symmetry result of Mariş can be improved as well, although foliated Schwarz symmetry should not be expected. We also note that assumption (P6) implies (P3) for $k = 2$, so we could have neglected assumption (P3) in Theorem 1.3.

In the following theorem, we summarize our results for the special class of systems (7).

Theorem 1.4. *Let P be given by (6) and suppose that (8) holds. Then $\mathcal{N} \subset \mathcal{M}$, and*

$$\inf_{\mathcal{N}} E = \inf_{u \in \mathcal{M}} \sup_{t_1, \dots, t_k \geq 0} E(t_1 u_1, \dots, t_k u_k) \quad (10)$$

is attained. Moreover, every minimizer $u \in \mathcal{N}$ of $E|_{\mathcal{N}}$ is a classical solution $u \in C^2(\Omega, \mathbb{R}^k) \cap C(\overline{\Omega}, \mathbb{R}^k)$ of (7). Moreover, if $k = 2$, Ω is a radial domain and V_1, V_2 are radial functions, then every ground state solution is such that u and v are foliated Schwarz symmetric with respect to antipodal points.

The paper is organized as follows. In Section 2 we will collect some preliminary results and give the proof of Theorem 1.1. In Section 3 we will prove Theorem 1.2 and therefore give – under additional assumptions – a minimax characterization of the value c defined in (3). The key result in this section is Proposition 3.2 below which will also be used in Section 4 where we prove our symmetry results for the special class of two-component systems (9). In particular, the proof of Theorem 1.3 is contained in Section 4. In Section 5 we consider special classes of systems satisfying our general assumptions. In particular, we will consider system (7) and prove Theorem 1.4 in this section. We close this section with an application of our symmetry results to a different setting.

2. SOME PRELIMINARIES AND THE EXISTENCE OF GROUND STATE SOLUTIONS

We will assume conditions (P0)–(P4) from now on, and we start with some general remarks on problem (1). First, by the transformation

$$\mathcal{H} \rightarrow \mathcal{H}, \quad u \mapsto \tilde{u} = (\sqrt{c_1} u_1, \dots, \sqrt{c_k} u_k) \quad (11)$$

problem (1) is reduced to the special case $c_1, \dots, c_k = 1$ with V_i replaced by $\frac{V_i}{c_i}$ and P replaced by

$$\tilde{P} \in \mathcal{C}(\mathbb{R}^k), \quad \tilde{P}(v_1, \dots, v_k) = P\left(\frac{v_1}{\sqrt{c_1}}, \dots, \frac{v_k}{\sqrt{c_k}}\right).$$

Moreover, the transformation (11) maps the corresponding Nehari sets and the sets \mathcal{M} into each other and preserves the value of the corresponding energy functionals. Hence we may assume from now on that

$$c_1 = \dots = c_k = 1.$$

As we will be interested in nonnegative solutions only, we will also assume from now on that

$$P(u_1, \dots, u_i, \dots, u_k) = P(u_1, \dots, |u_i|, \dots, u_k) \quad \text{for every } u \in \mathcal{H} \text{ and } i = 1, \dots, k. \quad (12)$$

Observe that this is consistent with the fact that $P \in C^2(\mathbb{R}^k)$ by the second assumption in (P2)). We then deduce from (P1) that E is a C^1 -functional on \mathcal{H} , and that

$$E(u_1, \dots, u_k) = E(|u_1|, \dots, |u_k|) \quad \text{for every } u \in \mathcal{H}. \quad (13)$$

In the following, we will write

$$\|u\|_i^2 := \int_{\Omega} (|\nabla u(x)|^2 + V_i(x)u^2) dx \quad \text{for } u \in H_0^1(\Omega)$$

and $i = 1, \dots, k$, and we note that the norms $\|\cdot\|_i$ are equivalent to the standard H_0^1 -norm as a consequence of assumption (P0). We also denote the L^p -norm by $\|\cdot\|_{L^p}$. We then define the Nehari manifold

$$\mathcal{N}_* := \left\{ u \in \mathcal{H} : u_i \not\equiv 0 \text{ and } \|u_i\|_i^2 = \int_{\Omega} P_{u_i}(u)u_i dx \text{ for } i = 1, \dots, k. \right\} \quad (14)$$

and we note that for $u = (u_1, \dots, u_k) \in \mathcal{H}$ we have the equivalence

$$u \in \mathcal{N}_* \quad \Longleftrightarrow \quad (|u_1|, \dots, |u_k|) \in \mathcal{N}$$

with \mathcal{N} defined in (2). Combining this with (13), we deduce that

$$\inf_{\mathcal{N}} E = \inf_{\mathcal{N}_*} E, \text{ so that every minimizer } u \in \mathcal{N} \text{ of } E|_{\mathcal{N}} \text{ also minimizes } E|_{\mathcal{N}_*}. \quad (15)$$

We now collect some easy consequences of assumptions (P1)–(P4).

Lemma 2.1. (i) *There exists $C > 0$ such that*

$$|P(u)| \leq C(1 + \sum_{i=1}^k |u_i|^p) \quad \text{and} \quad |P_{u_i}(u)u_i| \leq C(1 + \sum_{i=1}^k |u_i|^p) \quad \text{for all } u \in C^+. \quad (16)$$

(ii) *For every $u \in C^+$ we have $(2 + \alpha)P(u) \leq \sum_{i=1}^k P_{u_i}(u)u_i$.*

(iii) *$P_{u_i}(te_i)/t \rightarrow +\infty$ as $t \rightarrow +\infty$ for $i = 1, \dots, k$, where e_i denotes the i -th coordinate vector.*

Proof. (i) To see this, first take $\varphi_1(t) = P_{u_i}(tu)u_i$. We have, for $t \in [0, 1]$,

$$|\varphi_1'(t)| = \left| \sum_{j=1}^k P_{u_i u_j}(tu)u_i u_j \right| \leq \sum_{j=1}^k |P_{u_i u_j}(tu)u_i u_j| \leq \sum_{j=1}^k C|u_i||u_j|(1 + \sum_{i=1}^k |tu_i|^{p-2}) \leq C'(1 + \sum_{i=1}^k |u_i|^p)$$

which, together with the fact that $\varphi_1(0) = 0$, implies the upper bound for $|P_{u_i}(u)u_i|$. The same reasoning implies the other condition.

(ii) Consider the function $\varphi_2(t) = (2 + \alpha)P(tu) - \sum_{i=1}^k P_{u_i}(tu)tu_i$. For $t > 0$,

$$\begin{aligned} \varphi_2'(t) &= (2 + \alpha) \sum_{i=1}^k P_{u_i}(tu)u_i - \sum_{i,j=1}^k P_{u_i u_j}(tu)tu_i u_j - \sum_{i=1}^k P_{u_i}(tu)u_i \\ &= (1 + \alpha) \sum_{i=1}^k P_{u_i}(tu)u_i - \sum_{i,j=1}^k P_{u_i u_j}(tu)tu_i u_j \\ &= \frac{1}{t}(1, \dots, 1) \cdot M(tu) \cdot (1, \dots, 1)^T \leq 0. \end{aligned}$$

Then $\varphi_2(1) \leq \varphi_2(0) = 0$.

(iii) From (P4) one can see that $(1 + \alpha)P_{u_i}(te_i) \leq P_{u_i u_i}(te_i)te_i$, which implies (as $P_{u_i}(te_i) \neq 0$) that $P_{u_i}(te_i) \geq Ct^{1+\alpha}$ for some $C > 0$ and for every $t > 1$. \square

The remainder of this section will be devoted to the proof of Theorem 1.1. For this we need to study the sets \mathcal{N} and \mathcal{N}_* .

Lemma 2.2. *The Nehari manifold $\mathcal{N} \subset \mathcal{H}$ defined in (2) is nonempty.*

Proof. Take k functions $w_1, \dots, w_k \in H_0^1(\Omega)$ such that

$$w_i \neq 0, \quad w_i \geq 0 \text{ in } \Omega \text{ for } i = 1, \dots, k, \quad \text{and} \quad w_i \cdot w_j \equiv 0 \text{ in } \Omega \quad \text{whenever } i \neq j.$$

Consider the functions $\varphi_i : \mathbb{R}^k \mapsto \mathbb{R}$ defined by

$$\begin{aligned} \varphi_i(t_1, \dots, t_k) &= \|t_i w_i\|_i^2 - \int_{\Omega} P_{u_i}(t_1 w_1, \dots, t_i w_i, \dots, t_k w_k) t_i w_i \, dx \\ &= \|t_i w_i\|_i^2 - \int_{\{w_i > 0\}} P_{u_i}(0, \dots, t_i w_i, \dots, 0) t_i w_i \, dx. \end{aligned}$$

Observe first of all that (16) and (P2) imply the existence of a constant $C > 0$ such that

$$\int_{\Omega} P_{u_i}(0, \dots, t_i w_i, \dots, 0) t_i w_i \leq \frac{\|t_i w_i\|_i^2}{2} + C \|t_i w_i\|_i^p,$$

and hence

$$\varphi_i(t_1, \dots, t_k) \geq t_i^2 \left(\frac{\|w_i\|_i^2}{2} - C t_i^{p-2} \|w_i\|_i^p \right) > 0$$

for $t_i > 0$ very close to zero, uniformly in $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_k$. On the other hand, from Lemma 2.1-(iii) we see that

$$\varphi_i(t_1, \dots, t_k) = t_i^2 \left(\|w_i\|_i^2 - \int_{\{w_i > 0\}} \frac{P_{u_i}(0, \dots, t_i w_i, \dots, 0)}{t_i w_i} w_i^2 \, dx \right) \rightarrow -\infty$$

as $t_i \rightarrow +\infty$, uniformly in $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_k$. Hence we deduce the existence of $t_1, \dots, t_k > 0$ such that $\varphi_i(t_1, \dots, t_k) = 0 \, \forall i$, and $(t_1 w_1, \dots, t_k w_k) \in \mathcal{N}$, which is non empty. \square

Lemma 2.3. *There exists $\gamma > 0$ such that*

$$\|u_i\|_{L^p}, \|u_i\|_i \geq \gamma > 0 \quad \text{for every } u \in \mathcal{N}_*.$$

Proof. By using (P2), (P3) and (16), we know that for every $u \in \mathcal{N}_*$ we have

$$\begin{aligned} \|u_i\|_i^2 &= \int_{\Omega} P_{u_i}(u) u_i \, dx \leq \int_{\Omega} P_{u_i}(0, \dots, u_i, \dots, 0) u_i \, dx \\ &\leq \frac{1}{2} \|u_i\|_i^2 + C_1 \|u_i\|_{L^p}^p \leq \frac{1}{2} \|u_i\|_i^2 + C_2 \|u_i\|_i^p. \end{aligned}$$

Thus

$$\frac{1}{(2C_2)^{1/(p-2)}} \leq \|u_i\|_i, \quad \text{and} \quad \frac{1}{(2C_2)^{2/(p-2)}} \leq C_1 \|u_i\|_{L^p}^p.$$

\square

Lemma 2.4. *The set \mathcal{N}_* is a submanifold of \mathcal{H} of codimension k . Moreover, if $u \in \mathcal{N}_*$ is such that $E|'_{\mathcal{N}_*}(u) = 0$, then $E'(u) = 0$.*

Proof. The elements in \mathcal{N}_* are zeros of the functional $F : \mathcal{H} \rightarrow \mathbb{R}^k$, $u = (u_1, \dots, u_k) \mapsto (F_1(u), \dots, F_k(u))$ where, for each $i = 1, \dots, k$, F_i is the $C^1(\mathcal{H}, \mathbb{R})$ -functional defined by

$$F_i(u) = \|u_i\|_i^2 - \int_{\Omega} P_{u_i}(u) u_i \, dx.$$

Denote by \mathbf{T}_u the $k \times k$ matrix whose i -th line is the vector

$$F'(u)(0, \dots, u_i, \dots, 0) = (\partial_{u_i} F_1(u) u_i, \dots, \partial_{u_i} F_k(u) u_i).$$

Given $u \in \mathcal{N}_*$, for each i we have

$$\begin{aligned} \partial_{u_i} F_i(u) u_i &= 2\|u_i\|_i^2 - \int_{\Omega} (P_{u_i u_i}(u) u_i^2 + P_{u_i}(u) u_i) dx \\ &= -\alpha\|u_i\|_i^2 + (2 + \alpha)\|u_i\|_i^2 - \int_{\Omega} (P_{u_i u_i}(u) u_i^2 + P_{u_i}(u) u_i) dx \\ &= -\alpha\|u_i\|_i^2 + \int_{\Omega} ((1 + \alpha)P_{u_i}(u) u_i - P_{u_i u_i}(u) u_i^2) dx, \end{aligned}$$

while for $j \neq i$

$$\partial_{u_j} F_i(u) u_j = - \int_{\Omega} P_{u_i u_j}(u) u_i u_j dx.$$

Thus,

$$\mathbf{T}_{\mathbf{u}} = \left(-\alpha \delta_{ij} \|u_i\|_i^2 \right)_{i,j} + \left(\int_{\Omega} (\delta_{ij} (1 + \alpha) P_{u_i}(u) u_i - P_{u_i u_j}(u) u_i u_j) \right)_{i,j}$$

For every $\mathbf{z} \in \mathbb{R}^k$, we have that

$$\begin{aligned} \mathbf{z}^T \cdot \mathbf{T}_{\mathbf{u}} \cdot \mathbf{z} &= -\alpha \sum_{i=1}^k \|u_i\|_i^2 z_i^2 + \int_{\Omega} \mathbf{z}^T \cdot \left(\delta_{ij} (1 + \alpha) P_{u_i}(u) u_i - P_{u_i u_j}(u) u_i u_j \right)_{i,j} \cdot \mathbf{z} dx \\ &\leq -\alpha \sum_{i=1}^k \|u_i\|_i^2 z_i^2 \end{aligned}$$

by (P4), and hence $\mathbf{T}_{\mathbf{u}}$ is a negative definite matrix. In particular, its determinant is different from zero and the k vectors

$$F'(u)(u_1, 0, \dots, 0), \dots, F'(u)(0, \dots, 0, u_k)$$

are linearly independent. This implies that $F'(u) : \mathcal{H} \rightarrow \mathbb{R}^k$ is onto for every $u \in \mathcal{N}_*$, and hence \mathcal{N}_* is indeed a submanifold of \mathcal{H} of codimension k .

As for the second part of the lemma, if $J_{\beta}|'_{\mathcal{N}_*}(u) = 0$ then there exist real numbers λ_i , $i = 1, \dots, k$, such that $E'(u) = \sum_{i=1}^k \lambda_i F'_i(u)$. By testing the previous equality with $(0, \dots, 0, u_j, 0, \dots, 0)$, one obtains

$$0 = \sum_{i=1}^k \lambda_i \partial_{u_j} F_i(u) u_j, \quad \forall j = 1, \dots, k,$$

which is equivalent to

$$\mathbf{T}_{\mathbf{u}} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \mathbf{0}.$$

Hence $\lambda_i = 0$ for every i , and u is a critical point of the functional E . \square

Lemma 2.5. $E|_{\mathcal{N}_*}$ satisfies the Palais-Smale condition.

Proof. Here we recover the definitions of $F_i(u)$ and of $\mathbf{T}_{\mathbf{u}}$ from the proof of Lemma 2.4. Let $(u_n)_n = ((u_{1,n}, \dots, u_{k,n}))_n \subseteq \mathcal{N}_*$ be a Palais-Smale sequence for $E|_{\mathcal{N}_*}$, that is, $E(u_n)$ remains bounded in \mathbb{R} as $n \rightarrow \infty$ and

$$E'(u_n) - \sum_{i=1}^k \lambda_{i,n} F'_i(u_n) \rightarrow 0 \text{ in } \mathcal{H}', \text{ for some sequences } (\lambda_{1,n})_n, \dots, (\lambda_{k,n})_n \subset \mathbb{R}. \quad (17)$$

Observe that Lemma 2.1-(ii) implies that

$$\sum_{i=1}^k \|u_{i,n}\|_i^2 = \int_{\Omega} \sum_{i=1}^k P_{u_i}(u_n) u_{i,n} dx \geq (2 + \alpha) \int_{\Omega} P(u_n) dx,$$

whence

$$E(u_n) \geq \left(\frac{1}{2} - \frac{1}{2 + \alpha}\right) \sum_{i=1}^k \|u_{i,n}\|_i^2. \quad (18)$$

Thus $(u_n)_n$ is bounded in \mathcal{H} and (up to subsequences), we obtain the existence of $u = (u_1, \dots, u_k) \in \mathcal{H}$ such that

$$u_{i,n} \rightarrow u_i \text{ weakly in } H_0^1(\Omega), \text{ strongly in } L^q(\Omega) \text{ for every } 2 \leq q < 2^*.$$

By Lemma 2.3, we deduce that $u_i \not\equiv 0$ for every i . Moreover, we have

$$\mathbf{T}_{\mathbf{u}_n} = \left(\int_{\Omega} (\delta_{ij} P_{u_i}(u_n) u_{i,n} - P_{u_i u_i}(u_n) u_{i,n}^2) dx \right)_{i,j} \rightarrow \left(\int_{\Omega} (\delta_{ij} P_{u_i}(u) u_i - P_{u_i u_i}(u) u_i^2) dx \right)_{i,j} =: \mathbf{T}_{\mathbf{u}}$$

in \mathbb{R}^{2k} and, for each i ,

$$\|u_i\|_i^2 \leq \liminf_n \|u_{i,n}\|_i^2 = \liminf_n \int_{\Omega} P_{u_i}(u_n) u_{i,n} dx = \int_{\Omega} P_{u_i}(u) u_i dx.$$

Hence, by reasoning as in the proof of Lemma 2.4, we obtain that $\mathbf{T}_{\mathbf{u}}$ is a negative definite matrix, since $\forall \mathbf{z} \in \mathbb{R}^k$,

$$\begin{aligned} \mathbf{z}^T \cdot \mathbf{T}_{\mathbf{u}} \cdot \mathbf{z} &= -\alpha \sum_{i=1}^k z_i^2 \int_{\Omega} P_{u_i}(u) u_i dx + \int_{\Omega} \mathbf{z}^T \cdot \left(\delta_{ij} (1 + \alpha) P_{u_i}(u) u_i - P_{u_i u_j}(u) u_i u_j \right)_{i,j} \cdot \mathbf{z} dx \\ &\leq -\alpha \sum_{i=1}^k \|u_i\|_i^2 z_i^2. \end{aligned}$$

After testing (17) with $(0, \dots, u_{j,n}, \dots, 0)$ for every j , we obtain, as $n \rightarrow +\infty$,

$$\mathbf{o}(1) = \mathbf{T}_{\mathbf{u}_n} \cdot \begin{pmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{k,n} \end{pmatrix} = (\mathbf{T}_{\mathbf{u}} + \mathbf{o}(1)) \cdot \begin{pmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{k,n} \end{pmatrix}$$

and moreover

$$\begin{aligned} (\lambda_{1,n}, \dots, \lambda_{k,n}) \cdot \mathbf{o}(1) &= (\lambda_{1,n}, \dots, \lambda_{k,n}) \cdot \mathbf{T}_{\mathbf{u}} \cdot \begin{pmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{k,n} \end{pmatrix} + (\lambda_{1,n}, \dots, \lambda_{k,n}) \cdot \mathbf{o}(1) \cdot \begin{pmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{k,n} \end{pmatrix} \\ &\leq -C |(\lambda_{1,n}, \dots, \lambda_{k,n})|^2 + (\lambda_{1,n}, \dots, \lambda_{k,n}) \cdot \mathbf{o}(1) \cdot \begin{pmatrix} \lambda_{1,n} \\ \vdots \\ \lambda_{k,n} \end{pmatrix} \end{aligned}$$

for some $C > 0$. Thus for every i we have $\lambda_{i,n} \rightarrow 0$ and $\lambda_{i,n} F'_i(u_n) \rightarrow 0$ in \mathcal{H}' as $n \rightarrow \infty$, and therefore also $E'(u_n) \rightarrow 0$ in \mathcal{H}' as $n \rightarrow \infty$. By taking this time $(0, \dots, u_{i,n} - u_i, \dots, 0)$ as a test function, we obtain

$$E'(u_n)(0, \dots, u_{i,n} - u_i, \dots, 0) = o(1) \quad \text{as } n \rightarrow \infty,$$

which is equivalent to

$$\langle u_{i,n}, u_{i,n} - u_i \rangle_i - \int_{\Omega} P_{u_i}(u_n)(u_{i,n} - u_i) = o(1) \quad \text{as } n \rightarrow \infty.$$

Since $\int_{\Omega} P_{u_i}(u_n)(u_{i,n} - u_i) dx \rightarrow 0$, it follows that $\|u_{i,n}\|_i \rightarrow \|u_i\|_i$, which provides the strong convergence $u_{i,n} \rightarrow u_i$ for every i . \square

Proof of Theorem 1.1. We have that $c \geq 0$ (recall (18)). Let $(u_n)_n$ be a minimizing sequence for $E|_{\mathcal{N}_*}$, namely $u_n \in \mathcal{N}_*$ for all n and $E(u_n) \rightarrow \inf_{\mathcal{N}_*} E$ as $n \rightarrow \infty$. By the Ekeland's Variational Principle we can suppose, without loss of generality, that $(u_n)_n$ is a Palais-Smale sequence for the restricted functional $E|_{\mathcal{N}_*}$. Hence by Lemma 2.5 we have that, up to a subsequence, $u_n \rightarrow u$ strongly in \mathcal{H} . In particular $u \in \mathcal{N}_*$ (since $u_i \neq 0$ for all i by Lemma 2.3). Replacing u with $(|u_1|, \dots, |u_k|)$, we may assume that $u \in \mathcal{N}$, and by (15) we have $E(u) = c = \inf_{\mathcal{N}} E = \inf_{\mathcal{N}_*} E$. Moreover, u is a critical point of E by Lemma 2.4. As a consequence of the strong maximum principle, we then see that u is a solution of (1). \square

3. AN ALTERNATIVE CHARACTERIZATION FOR THE CRITICAL LEVEL c

This section is devoted to the proof of Theorem 1.2 and related facts. We will assume conditions (P0)–(P4) from now on. Given $u \in (H_0^1(\Omega) \setminus \{0\})^k$, we consider the function

$$\varphi = \varphi_u : \mathbb{R}^k \mapsto \mathbb{R}; \quad \varphi(t_1, \dots, t_k) := E(t_1 u_1, \dots, t_k u_k) = \sum_{i=1}^k \frac{t_i^2}{2} \|u_i\|_i^2 - \int_{\Omega} P(t_1 u_1, \dots, t_k u_k) dx.$$

We note that φ is even in each variable by (12). Moreover, the point $(0, \dots, 0)$ is always a strict local minimum for the function φ . In fact,

$$\varphi(t_1, \dots, t_k) = \frac{1}{2} \sum_{i=1}^k t_i^2 \|u_i\|_i^2 - \int_{\Omega} P(t_1 u_1, \dots, t_k u_k) dx \geq \frac{1}{2} \sum_{i=1}^k t_i^2 (\|u_i\|_i^2 - C t_i^{p-2} \|u_i\|_i^p) > 0$$

for sufficiently small $|t_1| + \dots + |t_k|$. Furthermore we observe that, if $t_1, \dots, t_k > 0$, then

$$(t_1 u_1, \dots, t_k u_k) \in \mathcal{N}_* \quad \Leftrightarrow \quad \nabla \varphi(t_1, \dots, t_k) = (0, \dots, 0).$$

Lemma 3.1. *Let $u \in \mathcal{H}$ with $u_i \neq 0$ for every i and take $t_1, \dots, t_k > 0$ such that $\nabla \varphi(t_1, \dots, t_k) = (0, \dots, 0)$. Then (t_1, \dots, t_k) is a non degenerate local maximum for φ .*

Proof. The proof follows some of the lines of the one of Lemma 2.4. If $(t_1, \dots, t_k) \in C^+$ is a critical point for φ , then

$$\|u_i\|_i^2 = \int_{\Omega} P_{u_i}(t_1 u_1, \dots, t_k u_k) \frac{t_i u_i}{t_i^2} dx \quad \text{for every } i,$$

and hence the Hessian matrix of φ at that point is given by

$$\begin{aligned} H_{\varphi}(t_1, \dots, t_k) &= \left(\frac{\partial^2 \varphi}{\partial t_i \partial t_j} \right)_{ij} = \left(\delta_{ij} \|u_i\|_i^2 - \int_{\Omega} P_{u_i u_j}(t_1 u_1, \dots, t_k u_k) u_i u_j dx \right)_{ij} \\ &= \left(-\alpha \delta_{ij} \|u_i\|_i^2 \right)_{ij} \\ &\quad + \left(\int_{\Omega} (\delta_{ij} (1 + \alpha) P_{u_i}(t_1 u_1, \dots, t_k u_k) \frac{t_i u_i}{t_i^2} - P_{u_i u_j}(t_1 u_1, \dots, t_k u_k) \frac{t_i u_i t_j u_j}{t_i t_j}) dx \right)_{ij}, \end{aligned}$$

which is negative definite, since for each $z \in \mathbb{R}^k$ we have

$$\begin{aligned} z^T \cdot H_\varphi(t_1, \dots, t_k) \cdot z &= -\alpha \sum_{i=1}^k \|u_i\|_i^2 z_i^2 + \int_{\Omega} \left(\frac{z_1}{t_1}, \dots, \frac{z_k}{t_k} \right) \cdot M(t_1 u_1, \dots, t_k u_k) \cdot \begin{pmatrix} \frac{z_1}{t_1} \\ \vdots \\ \frac{z_k}{t_k} \end{pmatrix} dx \\ &\leq -\alpha \sum_{i=1}^k \|u_i\|_i^2 z_i^2, \end{aligned}$$

where we have used (P4) in the last inequality. \square

We remark that assumption (P5) was not used in the proof above, but it will now allow us to control φ at the boundary of C^+ . The geometric meaning of (P5) can be formulated as follows. Fix u with $u_i \neq 0$ and consider $\psi(t) = E(u_1, \dots, t u_i, \dots, u_m)$. Then $\psi'(t) = 0$, and $\psi''(0) = \|u_i\|^2 - \int_{\Omega} P_{u_i u_i}(u_1, \dots, 0, \dots, u_k) u_i^2 dx > 0$. Hence (P5) implies that

$$E(u_1, \dots, t, \dots, u_k) > E(u_1, \dots, 0, \dots, u_k) \quad \text{for sufficiently small } t. \quad (19)$$

Proposition 3.2. *Let $u \in \mathcal{M}$. Then the function $\varphi = \varphi_u$ has precisely one critical point $(\bar{t}_1, \dots, \bar{t}_k)$ with $\bar{t}_1, \dots, \bar{t}_k > 0$. Moreover, φ_u attains a global maximum at this point, and $(\bar{t}_1 u_1, \dots, \bar{t}_k u_k) \in \mathcal{N}$.*

Proof. As $u \in \mathcal{M}$, we know that φ must have a global maximum at a point $\Lambda_0 = (\bar{t}_1, \dots, \bar{t}_k) \in C^+$. As the origin is a strict local minimum for φ and (19) holds, we must have $\bar{t}_i > 0$ for $i = 1, \dots, k$, and hence $(\bar{t}_1 u_1, \dots, \bar{t}_k u_k) \in \mathcal{N}$. Thus, by the previous lemma, Λ_0 is a non degenerate maximum. Suppose now, by contradiction, the existence of another critical point $\Lambda_1 \in C^+$ having only positive components. By Lemma 3.1, both Λ_0 and Λ_1 are nondegenerate local maxima of φ . Hence for

$$\bar{c} = \sup_{A \in \bar{\Gamma}} \min_A \varphi, \quad \text{where } \bar{\Gamma} = \{A \subseteq \mathbb{R}^k : A \text{ is compact, connected, and } \Lambda_0, \Lambda_1 \in A\},$$

we have $\bar{c} < \min \varphi(\Lambda_0), \varphi(\Lambda_1)$. The class $\bar{\Gamma}$ was already considered in the paper [2]. We will now show the existence of a optimal set in $\bar{\Gamma}$, which contains a critical point of φ at level \bar{c} . This idea is inspired by the work [5]. Define

$$K_{\bar{c}} = \{(t_1, \dots, t_k) \in \mathbb{R}^k : \varphi(t_1, \dots, t_k) = \bar{c} \text{ and } \nabla \varphi(t_1, \dots, t_k) = (0, \dots, 0)\}.$$

Since $u \in \mathcal{M}$, there exists $R > 0$ such that

$$\varphi(t_1, \dots, t_k) < \bar{c} - 1 \quad \text{if } |t_1|^2 + \dots + |t_k|^2 \geq R^2. \quad (20)$$

Let us now put

$$B := B_R(0) \cap C^+ \quad \text{and} \quad B_\varepsilon := \{t \in B_R(0) : t_i \geq \varepsilon \text{ for } i = 1, \dots, k\} \subset B$$

for every $\varepsilon > 0$. As a consequence of (19) and since 0 is a strict local minimum of φ , for $\varepsilon > 0$ sufficiently small there exists a map $\psi : B \rightarrow B_\varepsilon$ such that $\varphi(\psi(t)) \geq \varphi(t)$ for every $t \in B$ and $\psi(t) = t$ for every $t \in B_\varepsilon$. We fix ε and ψ with this property and such that $\Lambda_0, \Lambda_1 \in B_\varepsilon$. We now claim:

1. There exists $A_* \in \bar{\Gamma}$ such that $A_* \subset B_\varepsilon$ and $\min_{A_*} \varphi = \bar{c}$.

To prove this, take a maximizing sequence for \bar{c} , namely $A_n \in \bar{\Gamma}$ such that $\bar{c} - 1/n \leq \min_{A_n} \varphi \leq$

c. By (20), we then have $A_n \subset B_R(0)$ for every $n \in \mathbb{N}$. Therefore the set

$$A_* := \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} A_i} \subset B_R(0)$$

is compact and connected, and $\Lambda_0, \Lambda_1 \in \bar{A}$. Moreover, $\bar{c} \leq \min_{A_*} \varphi$. Therefore $A_* \in \bar{\Gamma}$ and $\min_{A_*} \varphi = \bar{c}$. As φ is even with respect to each coordinate, we can suppose without loss of generality that $A_* \subseteq C^+$ and hence $A_* \subset B$. Moreover, replacing A_* by $\psi(A_*)$ if necessary and recalling that $\psi(\Lambda_i) = \Lambda_i$ by our choice of ε and ψ , we may assume that $A_* \subset B_\varepsilon$.

2. $A_* \cap K_{\bar{c}} \neq \emptyset$.

Suppose this is not true. Then, by the deformation lemma [20, Theorem 3.4], there exists a neighborhood \mathcal{V} of $K_{\bar{c}}$ such that $A_* \cap \mathcal{V} = \emptyset$, $\varepsilon < (\varphi(\Lambda_0) - \bar{c})/2$, and a homeomorphism $h : \mathbb{R}^k \mapsto \mathbb{R}^k$ such that

- $h(t_1, \dots, t_k) = (t_1, \dots, t_k)$, $|\varphi(t_1, \dots, t_k) - \bar{c}| \geq 2\varepsilon$;
- $\varphi(h(t_1, \dots, t_k)) \geq \bar{c} + \varepsilon$ for every $(t_1, \dots, t_k) \notin \mathcal{V}$ such that $\varphi(t_1, \dots, t_k) \geq \bar{c} - \varepsilon$.

Observe that $h(A_*)$ is a compact and connected set. Moreover, $\varphi(\Lambda_0) = \varphi(\Lambda_1) > \bar{c} + 2\varepsilon$, then $h(\Lambda_0) = u_0, h(\Lambda_1) = u_1$ and $\Lambda_0, \Lambda_1 \in h(A_*)$. Hence $h(A_*) \in \bar{\Gamma}$, and

$$\bar{c} + \varepsilon \leq \min_{h(A_*)} \varphi \leq \bar{c},$$

which is a contradiction. Hence $A_* \cap K_{\bar{c}} \neq \emptyset$, as claimed.

Now, to reach a final contradiction, let $t = (t_1, \dots, t_k) \in A_* \cap K_{\bar{c}}$. Since $t_i \geq \varepsilon$ for every i , we deduce from Lemma 3.1 that t is a strict local maximum of φ . Since A_* is connected, this however implies that $\min_{A_*} \varphi < \varphi(t) = \bar{c}$, which contradicts 1. above. \square

Proof of Theorem 1.2. Let $u \in \mathcal{M}$. By Proposition 3.2 there exists $(\bar{t}_1, \dots, \bar{t}_k)$ such that $(\bar{t}_1 u_1, \dots, \bar{t}_k u_k) \in \mathcal{N}$ and such that $(\bar{t}_1, \dots, \bar{t}_k)$ is a maximum for φ in C^+ . Hence

$$c = \inf_{\mathcal{N}} E \leq E(\bar{t}_1 u_1, \dots, \bar{t}_k u_k) \leq \sup_{t_1, \dots, t_k \geq 0} E(t_1 u_1, \dots, t_k u_k)$$

and this shows (5). Moreover, if $u \in \mathcal{M}$ for some minimizer $u \in \mathcal{N}$ of $E|_{\mathcal{N}}$, then $(1, \dots, 1)$ is a critical point of φ_u and therefore a global maximum of φ by Proposition 3.2. Hence

$$\sup_{t_1, \dots, t_k \geq 0} E(t_1 u_1, \dots, t_k u_k) = E(u) = c,$$

and therefore equality holds in (5). \square

4. A GENERAL SYMMETRY RESULT FOR THE CASE OF TWO EQUATIONS

Here we will restrict our attention to the two component system (9). By the arguments in the beginning of Section 2, we may assume that $c_1 = c_2 = 1$, so we are dealing with the system

$$-\Delta u + V_1(x)u = P_u(u, v) \quad -\Delta v + V_2(x)u = P_v(u, v) \quad u, v \in H_0^1(\Omega). \quad (21)$$

We suppose from now on that Ω is a radial domain, namely a ball or an annulus, and that V_1 and V_2 are radial functions, i.e. $V_i(x) = V_i(y)$ for all $x, y \in \Omega$ with $|x| = |y|$ and $i = 1, 2$. As already remarked in the introduction, we cannot expect ground state solutions of (21) to be radial (see Remark 5.4 below for a counterexample). However, via polarization methods we will show Theorem 1.3 which states that under the “negative coupling assumption” (P6) ground state solutions are foliated Schwarz symmetric (as defined in the introduction) in each of their components with respect to antipodal points. We will state an abstract criterion

for this type of symmetry of solutions of (21) first (see Theorem 4.3). This criterion is of independent interest and has applications within a different setting, see Subsection 5.1 below. Let us introduce some useful notations. We define the sets

$$\mathcal{H}_0 = \{H \subset \mathbb{R}^N : H \text{ is a closed half-space in } \mathbb{R}^N \text{ and } 0 \in \partial H\}$$

and, for $p \neq 0$,

$$\mathcal{H}_0(p) = \{H \in \mathcal{H}_0 : p \in \text{int}(H)\}.$$

For each $H \in \mathcal{H}_0$ we denote by $\sigma_H : \mathbb{R}^N \rightarrow \mathbb{R}^N$ the reflection in \mathbb{R}^N with respect to the hyperplane ∂H , and define the polarization of a function $u : \Omega \rightarrow \mathbb{R}$ with respect to H by

$$u_H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\} & x \in H \cap \Omega, \\ \min\{u(x), u(\sigma_H(x))\} & x \in \Omega \setminus H. \end{cases}$$

Moreover, we will call $H \in \mathcal{H}_0$ *dominant* for u if $u(x) \geq u(\sigma_H(x))$ for all $x \in \Omega \cap H$ (or, equivalently, $u_H(x) = u(x)$ for every $x \in \Omega \cap H$). On the other hand we will say that $H \in \mathcal{H}_0$ is *subordinate* for u if $u(x) \leq u(\sigma_H(x))$ for all $x \in \Omega \cap H$. We recall from [4, Lemma 4.2] (see also [24, Proposition 2.7]) the following characterization of foliated Schwarz symmetry.

Proposition 4.1. *Let $u : \Omega \rightarrow \mathbb{R}$ be a continuous function. Then u is foliated Schwarz symmetric with respect to $p \in \partial B_1(0)$ if and only if every $H \in \mathcal{H}_0(p)$ is dominant for u .*

Moreover, we will need the following properties (see for instance [24, Lemma 3.1]).

Lemma 4.2. *Let $u : \Omega \rightarrow \mathbb{R}$ be a measurable function and $H \in \mathcal{H}_0$.*

- (i) *If $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $F(x, t) = F(y, t)$ for every $x, y \in \Omega$ such that $|x| = |y|$ and $t \in \mathbb{R}$ and $\int_{\Omega} |F(x, u(x))| dx < +\infty$, then $\int_{\Omega} F(x, u_H) dx = \int_{\Omega} F(x, u) dx$.*
- (ii) *Moreover, if $u \in H_0^1(\Omega)$ then also $u_H \in H_0^1(\Omega)$ and $\int_{\Omega} |\nabla u_H|^2 = \int_{\Omega} |\nabla u|^2$.*

For every $H \in \mathcal{H}_0$ we denote by $\widehat{H} \in \mathcal{H}_0$ the closure of the complementary half-space $\mathbb{R}^N \setminus H$. We can now state the main abstract result of this section.

Theorem 4.3. *Take $P \in C^2(\mathbb{R}^2)$ such that*

$$(P6) \quad P_{uv}(s, t) < 0 \text{ for every } s, t > 0.$$

Let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a classical solution of (21). If, for every $H \in \mathcal{H}_0$, the pair $(u_H, v_{\widehat{H}})$ is also a strong solution of (21), then u and v are foliated Schwarz symmetric with respect to antipodal points, that is, there exists $p \in \partial B_1(0)$ such that u is foliated Schwarz symmetric with respect to p , and v is foliated Schwarz symmetric with respect to $-p$.

Proof. Take $r > 0$ such that $\partial B_r(0) \subseteq \Omega$ and let $p \in \partial B_1(0)$ be such that $\max_{\partial B_r(0)} u = u(rp)$. Given $H \in \mathcal{H}_0(p)$, we will prove that H is dominant for u and subordinate for v . This combined with Proposition 4.1 immediately provides the conclusion of the theorem. From

$$-\Delta u + V_1(x)u = P_u(u, v), \quad \text{and} \quad -\Delta u_H + V_1(x)u_H = P_u(u_H, v_{\widehat{H}})$$

it follows that, for $x \in \Omega \cap H$, $w(x) := u_H(x) - u(x) \geq 0$ and

$$-\Delta w + c(x)w = P_u(u_H, v_{\widehat{H}}) - P_u(u_H, v), \quad (22)$$

with $c(x) = V_1(x) - (P_u(u_H, v) - P_u(u, v))/(u_H - u) \in L_{\text{loc}}^\infty(\Omega)$. As $v_{\widehat{H}} \leq v$ in $\Omega \cap H$, condition (P6) implies that $P_u(u_H, v) \leq P_u(u_H, v_{\widehat{H}})$ in $\Omega \cap H$. Thus

$$-\Delta w + c(x)w \geq 0 \quad \text{and} \quad w \geq 0 \quad \text{in } \Omega \cap H,$$

which implies (by the Strong Maximum Principle, see for instance [10, Theorem 1.7]) that either $w > 0$ or $w \equiv 0$ in $\Omega \cap H$. By the choice of p , we have that $rp \in \Omega \cap H$ and that $w(rp) = 0$, and then it must be $u = u_H$ and therefore $w \equiv 0$ in $\Omega \cap H$. Moreover, coming back to (22), we now see that

$$P_u(u_H, v_{\widehat{H}}) = P_u(u_H, v) \quad (23)$$

and hence, since the map $t \mapsto P_u(s, t)$ is strictly decreasing for each fixed s as a consequence of (P6), we obtain $v = v_{\widehat{H}}$ in $\Omega \cap H$. Thus we have proved that H is dominant for u and subordinate for v , and the theorem follows. \square

Remark 4.4. First we observe that (P6) implies condition (P3). Second, we note that Theorem 4.3 holds true under slightly more general assumptions replacing (P6): we can assume instead that

for each $s \geq 0$, the function $t \mapsto P_u(s, t)$ is nonincreasing in $[0, \infty)$ and strictly decreasing in $[0, \varepsilon)$ for some $\varepsilon > 0$.

In fact, one can proceed in the previous proof until (23). Then, by looking at the second equations of the systems, we would have

$$-\Delta(v - v_{\widehat{H}}) + \left(V_2(x) - \frac{P_v(u, v) - P_v(u, v_{\widehat{H}})}{v - v_{\widehat{H}}} \right) (v - v_{\widehat{H}}) = 0 \text{ and } v \geq v_{\widehat{H}} \text{ in } \Omega \cap H,$$

which gives that either $v > v_{\widehat{H}}$ or $v = v_{\widehat{H}}$ in $\Omega \cap H$. Thus by (23) and the new assumptions we would have equality.

Alternatively, we could have also supposed that

for each $t \geq 0$, the function $s \mapsto P_v(s, t)$ is nonincreasing in $[0, \infty)$ and strictly decreasing in $[0, \varepsilon)$ for some $\varepsilon > 0$.

Before we may complete the proof of Theorem 1.3, we first need the following lemma.

Lemma 4.5. *Let $P \in C^2(\mathbb{R}^2)$ be such that (P6) holds. Take $u, v > 0$ such that $\int_{\Omega} P(u, v) dx < +\infty$. Then for every $H \in \mathcal{H}_0$ we have that*

$$\int_{\Omega} P(u, v) dx \leq \int_{\Omega} P(u_H, v_{\widehat{H}}) dx.$$

Proof. We claim that

$$P(a, c) + P(b, d) \leq P(\max\{a, b\}, \min\{c, d\}) + P(\min\{a, b\}, \max\{c, d\})$$

for every $a, b, c, d > 0$. In the case that $a \geq b$ and $d \geq c$ the result trivially holds. On the other hand, suppose that $a \geq b$ and $c \geq d$. Then

$$\begin{aligned} 0 &\geq \int_b^a \int_d^c P_{uv}(\xi, \zeta) d\zeta d\xi = \int_b^a (P_u(\xi, c) - P_u(\xi, d)) d\xi \\ &= P(a, c) - P(b, c) - P(a, d) + P(b, d), \end{aligned}$$

which proves the claim. From this we conclude that

$$\begin{aligned}
\int_{\Omega} P(u, v) dx &= \int_{\Omega \cap H} [P(u(x), v(x)) + P(u(\sigma_H(x)), v(\sigma_H(x)))] dx \\
&\leq \int_{\Omega \cap H} [P(u_H(x), v_{\widehat{H}}(x)) + P(u_H(\sigma_H(x)), v_{\widehat{H}}(\sigma_H(x)))] dx \\
&= \int_{\Omega} P(u_H, v_{\widehat{H}}) dx.
\end{aligned}$$

□

Finally we may complete the

Proof of Theorem 1.3. Let $(u, v) \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2)$ be a classical solution of (9) minimizing $E|_{\mathcal{N}}$ and such that $(u, v) \in \mathcal{M}$. Take $H \in \mathcal{H}_0$. By Theorem 4.3, we only need to show that $(u_H, v_{\widehat{H}})$ is also a solution to (21). First of all observe that for each $t, s > 0$, we have

$$\begin{aligned}
E(tu_H, sv_{\widehat{H}}) &= \frac{t^2}{2} \|u_H\|_1^2 + \frac{s^2}{2} \|v_{\widehat{H}}\|_2^2 - \int_{\Omega} P(tu_H, sv_{\widehat{H}}) dx \\
&= \frac{t^2}{2} \|u_H\|_1^2 + \frac{s^2}{2} \|v_{\widehat{H}}\|_2^2 - \int_{\Omega} P((tu)_H, (sv)_{\widehat{H}}) dx \\
&\leq \frac{t^2}{2} \|u\|_1^2 + \frac{s^2}{2} \|v\|_2^2 - \int_{\Omega} P(tu, sv) dx \\
&= E(tu, sv),
\end{aligned}$$

where we have used Lemmas 4.2 and 4.5. Hence, as $(u, v) \in \mathcal{M}$, we have that also $(u_H, v_{\widehat{H}}) \in \mathcal{M}$, and so there exists $\bar{t}, \bar{s} > 0$ such that $(\bar{t}u_H, \bar{s}v_{\widehat{H}}) \in \mathcal{N}$. Therefore, by Proposition 3.2,

$$c \leq E(\bar{t}u_H, \bar{s}v_{\widehat{H}}) \leq E(\bar{t}u, \bar{s}v) \leq \max_{t, s \geq 0} E(tu, sv) = E(u, v) = c.$$

and thus $\bar{t} = \bar{s} = 1$ by the uniqueness of the maximum as stated in Proposition 3.2. Thus $(u_H, v_{\widehat{H}}) \in \mathcal{N}$ and $E(u_H, v_{\widehat{H}}) = c$. Therefore the second statement in Theorem 1.1 implies that $(u_H, v_{\widehat{H}})$ is a solution of (21), as required. □

5. SOME SPECIAL SYSTEM CLASSES

In this section, we will discuss results for special subclasses of system (1), and in particular we will give the proof of Theorem 1.4 which is concerned with problem (7). Motivated in particular by results in the papers [7, 8], we now discuss a general family of functions P where the interaction terms are separated from the others. For this let $H \in C^2(\mathbb{R}^k)$ and $f_i \in C^1(\mathbb{R})$ for $i = 1, \dots, k$. Define $F_i(s) := \int_0^s f_i(\xi) d\xi$. For

$$P \in C^2(\mathbb{R}^k), \quad P(u) = \sum_{i=1}^k F_i(u_i) - H(u), \quad (24)$$

let us see under which assumptions P satisfies (P1)–(P4). We consider the following assumptions for the functions f_i .

(a1) For each i there exists a constant $C_i > 0$ such that

$$|f'_i(s)| \leq C_i(1 + |s|^{p-2}) \quad \text{for } s \geq 0 \text{ with some } p \in (2, 2^*),$$

where $2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = +\infty$ otherwise.

(a2) $f_i(s) = o(s)$ as $s \rightarrow 0$, for every $i = 1, \dots, k$.

(a3) There exists $\gamma > 0$ ($2 + \gamma \leq p$) such that

$$0 < (1 + \gamma)f_i(s)s \leq f_i'(s)s^2, \quad \text{for all } s \geq 0.$$

Moreover, for the interaction potential H we assume the following.

(H1) There exist constants $C > 0$ and $0 < \alpha \leq \gamma$ such that

$$|H_{u_i u_j}(u)| \leq C(1 + \sum_{i=1}^k |u_i|^\alpha), \quad \text{for } i, j \in \{1, \dots, k\}, u \in C^+.$$

(H2) $H(0) = 0$ and $H_{u_i}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_k) = 0$ for $i = 1, \dots, k$ and $u \in C^+$.

(H3) $H_{u_i}(u) \geq 0$ for $i = 1, \dots, k$ and $u \in C^+$.

(H4) For every $u \in C^+$, the matrix

$$(h_{ij})_{ij} = \left(\delta_{ij}(1 + \alpha)H_{u_i}(u)u_i - H_{u_i u_j}(u)u_i u_j \right)_{i,j=1,\dots,k}$$

is positive semidefinite, where α is the constant appearing on (H1).¹

We then have the following result.

Theorem 5.1. *Let f_i satisfy (a1)–(a3) and H satisfy (H1)–(H4). Then (P1)–(P5) hold for P defined in (24). Hence, if the functions $V_i \in L^\infty(\Omega)$, $i = 1, \dots, k$, satisfy (P0), then the assertions of Theorems 1.1 and 1.2 are true. In particular, the system*

$$\begin{cases} -\Delta u_i + V_i(x)u = f_i(u_i) - H_{u_i}(u) & i = 1, \dots, k. \\ u_i \in H_0^1(\Omega)(u), u_i > 0 \text{ in } \Omega. \end{cases}$$

admits a non-trivial solution which minimizes the functional $E|_{\mathcal{N}}$.

Moreover, if in addition $\alpha < \gamma$ in (H1), then every $u \in \mathcal{H}$ with $u_i \geq 0, u_i \not\equiv 0$ for $i = 1, \dots, k$ is contained in \mathcal{M} , and therefore equality holds in (5).

Proof. (P1) is an immediate consequence of (a1) and (H1), and (P2) is an immediate consequence of (a2) and (H2). (P3) follows directly from (H3), and (P4) follows directly from (a3) and (H4). As for (P5), observe that

$$P_{u_i u_i}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_k) = f_i'(0) - \lim_{t \rightarrow 0^+} \frac{H_{u_i}(u_1, \dots, u_{i-1}, t, u_{i+1}, \dots, u_k)}{t} \leq f_i'(0) = 0,$$

for $i = 1, \dots, k$ by (a2) and (H3). As a consequence, the assertions of Theorems 1.1 and 1.2 are true.

Finally, let us assume that $\alpha < \gamma$ holds in assumption (H1), and let $u \in \mathcal{H}$ with $u_i \geq 0, u_i \not\equiv 0$ for $i = 1, \dots, k$. We show that $u \in \mathcal{M}$. For this we note that condition (a3) implies the existence of constants $C_i, D_i > 0$ such that

$$F_i(t) \geq C_i t^{2+\gamma} - D_i, \quad \forall s \geq 0, i = 1, \dots, k.$$

Thus, for some constant $C_1 > 0$,

$$\begin{aligned} E(t_1 u_1, \dots, t_k u_k) &= \sum_{i=1}^k \left(\frac{t_i^2}{2} \|u_i\|_i^2 - \int_{\Omega} F_i(t_i u_i) dx \right) + \int_{\Omega} H(t_1 u_1, \dots, t_k u_k) dx \\ &\leq \sum_{i=1}^k \left(\frac{t_i^2}{2} \|u_i\|_i^2 - C_i t_i^{2+\gamma} \int_{\Omega} |u_i|^{2+\gamma} dx + C t_i^{2+\alpha} \int_{\Omega} |u_i|^{2+\alpha} dx \right) + C_1 \rightarrow -\infty, \end{aligned}$$

¹Actually this is equivalent to ask (H1) and (H4) for two different constants $\alpha_1, \alpha_2 \leq \gamma$, as in each case if each assumption is true for some β , it is true for every $\tilde{\beta} \geq \beta$.

as $|t_1| + \dots + |t_k| \rightarrow +\infty$, since $\gamma > \alpha$ and $\int_{\Omega} |u_i|^{2+\gamma} dx > 0$ for $i = 1, \dots, k$. \square

Theorem 5.1 generalizes the existence result contained in [7, Theorem 2.1] and [8, Theorem 2.2]. The main difference is that we allow $\alpha = \gamma$ in (H1), which means that we allow F_i and H to have the same kind of growth at infinity. We point out that in this case it is not necessarily true that $u \in \mathcal{M}$ for every $u \in \mathcal{H}$ with $u_i \geq 0, u_i \not\equiv 0$ for $i = 1, \dots, k$. As an example, consider the two-component system

$$-\Delta u_1 = u_1^3 - \beta u_2^2 u_2, \quad -\Delta u_2 = u_2^3 - \beta u_1^2 u_1 \quad u_1, u_2 \in H_0^1(\Omega) \quad (25)$$

which in dimension $N \leq 3$ and for $\beta > 0$ is a special case of assumptions (a1)–(a3), (H1)–(H4) with $f_1(t) = f_2(t) = t^3$, $H(u_1, u_2) = \frac{\beta}{2} u_1^2 u_2^2$ and $\alpha = \gamma = 2$. The corresponding energy functional is then given by

$$u = (u_1, u_2) \mapsto E(u) = \sum_{i=1}^2 \int_{\Omega} \left(\frac{1}{2} |\nabla u_i|^2 - \frac{1}{4} |u_i|^4 \right) dx + \frac{\beta}{2} \int_{\Omega} |u_1|^2 |u_2|^2 dx,$$

and in case $\beta \geq 1$ we have $E(tw, tw) \rightarrow +\infty$ as $t \rightarrow \infty$ for every $w \in H_0^1(\Omega) \setminus \{0\}$, so that $(w, w) \notin \mathcal{M}$. Nevertheless, we will be able to show $\mathcal{N} \subset \mathcal{M}$ for system (25) and the more general class of systems (7) arising from the choice of functions

$$f_i(u) = \lambda_i u^{p-1} \quad \text{and} \quad H(u) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^k \beta_{ij} u_i^{q_i} u_j^{q_j}, \quad (26)$$

where $2 < p < 2^*$ and the other parameters satisfy (8). This also leads to equality in (5) and therefore to a minimax characterization of $\inf_{\mathcal{N}} E$.

Proposition 5.2. *The class of functions given by (26) satisfies assumptions (a1)–(a3) and (H1)–(H4) with $\alpha = \gamma = p - 2$. Moreover, if the functions $V_i \in L^\infty(\Omega)$, $i = 1, \dots, k$, satisfy (P0), then we have $\mathcal{N} \subset \mathcal{M}$ in this case, where \mathcal{N} and \mathcal{M} are defined with respect to the corresponding functional*

$$u \mapsto E(u) = \frac{1}{2} \sum_{i=1}^k \|u_i\|_i^2 - \sum_{i=1}^k \frac{\lambda_i}{p} \int_{\Omega} |u_i|^p dx + \sum_{i \neq j} \beta_{ij} \int_{\Omega} |u|^{q_i} |v|^{q_j} dx.$$

Proof. Assumptions (a1)–(a3) and (H1)–(H3) are rather immediate. We now show that also (H4) holds with the choice $\alpha = p - 2$. Let $u \in C^+$ and recall the matrix $(h_{ij})_{ij}$ defined in (H4). We have for each i

$$h_{ii} = (1 + \alpha) H_{u_i}(u) u_i - H_{u_i u_i}(u) u_i^2 = (p - q_i) q_i u_i^{q_i} \sum_{j \neq i} \beta_{ij} u_j^{q_j} > 0$$

and, for $j \neq i$,

$$h_{ij} = -H_{u_i u_j}(u) u_i u_j = -q_i q_j \beta_{ij} u_i^{q_i} u_j^{q_j} < 0.$$

By the Gershgorin's theorem (see for instance [11, Appendix 7]), the eigenvalues of $(h_{ij})_{ij}$ lie in the set

$$\begin{aligned} \bigcup_{i=1}^k \left\{ \lambda : |\lambda - h_{ii}| \leq \sum_{j \neq i} |h_{ij}| \right\} &\subseteq \bigcup_{i=1}^k \left\{ \lambda : \lambda \geq h_{ii} + \sum_{j=1}^k h_{ij} \right\} \\ &= \bigcup_{i=1}^k \left\{ \lambda : \lambda \geq \sum_{j \neq i} \beta_{ij} q_i (p - q_i - q_j) u_i^{q_i} u_j^{q_j} \right\}. \end{aligned}$$

Hence (8) implies that all eigenvalues of $(h_{ij})_{ij}$ are nonpositive, and hence $(h_{ij})_{ij}$ is a negative semidefinite matrix.

To show that $\mathcal{N} \subset \mathcal{M}$, let $u \in \mathcal{N}$. For $(t_1, \dots, t_k) \in C^+$, we then have, by Young's inequality,

$$t_i^{q_i} t_j^{q_j} = \frac{q_i}{p} t_i^p + \frac{q_j}{p} t_j^p + \kappa_{ij} \quad \text{for } i, j = 1, \dots, k \text{ with } \kappa_{ij} = 1 - \frac{q_i + q_j}{p} \geq 0.$$

Consequently,

$$\begin{aligned} E(t_1 u_1, \dots, t_k u_k) &= \frac{1}{2} \sum_{i=1}^k t_i^2 \|u_i\|_i^2 - \sum_{i=1}^k \frac{\lambda_i}{p} t_i^p \int_{\Omega} |u_i|^p dx + \sum_{i \neq j} t_i^{q_i} t_j^{q_j} \beta_{ij} \int_{\Omega} |u|^{q_i} |v|^{q_j} dx \\ &\leq \frac{1}{2} \sum_{i=1}^k t_i^2 \|u_i\|_i^2 - \sum_{i=1}^k \frac{\lambda_i}{p} t_i^p \int_{\Omega} |u_i|^p dx + \sum_{i \neq j} \left(\frac{q_i t_i^p + q_j t_j^p}{p} + \kappa_{ij} \right) \beta_{ij} \int_{\Omega} |u|^{q_i} |v|^{q_j} dx \\ &= \frac{1}{2} \sum_{i=1}^k t_i^2 \|u_i\|_i^2 - \sum_{i=1}^k \frac{t_i^p}{p} \left(\lambda_i \int_{\Omega} |u_i|^p dx - q_i \sum_{j \neq i} \beta_{ij} \int_{\Omega} |u|^{q_i} |v|^{q_j} dx \right) + \kappa \\ &= \sum_{i=1}^k \left(\frac{t_i^2}{2} - \frac{t_i^p}{p} \right) \|u_i\|_i^2 + \kappa \rightarrow -\infty \quad \text{as } t_1 + \dots + t_k \rightarrow +\infty \end{aligned}$$

with $\kappa = \sum_{i \neq j} \kappa_{ij} \beta_{ij} \int_{\Omega} |u|^{q_i} |v|^{q_j} dx$, where in the last step we have used that $u \in \mathcal{N}$. This shows $u \in \mathcal{M}$, and we conclude that $\mathcal{N} \subset \mathcal{M}$. \square

We may now complete the

Proof of Theorem 1.4. By Proposition 5.2 and Theorem 5.1, assumptions (P1)-(P5) are satisfied for P given in (6). Hence Theorem 1.1 implies that $\inf_{\mathcal{N}} E$ is attained, and that every minimizer $u \in \mathcal{N}$ of $E|_{\mathcal{N}}$ is a weak solution of (7). Moreover, by elliptic regularity, noting that the right hand side of (21) is Hölder continuous, we find that $u \in C^2(\Omega, \mathbb{R}^k) \cap C(\overline{\Omega}, \mathbb{R}^k)$ is in fact a classical solution. Since we also know from Proposition 5.2 that $\mathcal{N} \subset \mathcal{M}$, Theorem 1.2 implies that

$$\inf_{\mathcal{N}} E = \inf_{u \in \mathcal{M}} \sup_{t_1, \dots, t_k \geq 0} E(t_1 u_1, \dots, t_k u_k),$$

and in case $k = 2$ with Ω, V_1, V_2 radially symmetric, it follows from Theorem 1.3 that every $u \in C^2(\Omega, \mathbb{R}^k) \cap C(\overline{\Omega}, \mathbb{R}^k)$ minimizing E on \mathcal{N} is such that u and v are foliated Schwarz symmetric with respect to antipodal points. \square

We add a symmetry result corresponding to the class of functions (24) in the case $k = 2$ under the extra assumption that $\alpha < \gamma$ in (H1). Hence we consider a system of the type

$$\begin{cases} -\Delta u = f_1(u) - H_u(u, v), \\ -\Delta v = f_2(v) - H_v(u, v) \\ u, v \in H_0^1(\Omega), \quad u, v > 0 \text{ in } \Omega. \end{cases} \quad (27)$$

Theorem 5.3. *Take f_1, f_2 satisfying (a1)-(a3) and H satisfying $(\tilde{H}1)$, (H2)-(H4) and (H5) $H_{uv}(s, t) > 0$ for every $s, t > 0$.*

Furthermore, suppose that Ω is radially symmetric, and that $V_1, V_2 \in L^\infty(\Omega)$ are radial functions satisfying (P0). Let $(u, v) \in \mathcal{N}$ be a minimizer of $E|_{\mathcal{N}}$. Then u and v are foliated Schwarz symmetric with respect to antipodal points.

Proof. This is a direct consequence of Theorem 1.3, since the second statement of Theorem 5.1 implies that $(u, v) \in \mathcal{M}$ as a consequence of assumption $(\tilde{H}1)$. \square

Remark 5.4. In general, minimal energy solution to (27) are not radial. So see this, let us rewrite the system (27) with an extra parameter $\beta > 0$

$$\begin{cases} -\Delta u = f(u) - \beta H_u(u, v), \\ -\Delta v = f(v) - \beta H_v(u, v) \\ u, v \in H_0^1(\Omega), \quad u, v > 0 \text{ in } \Omega. \end{cases} \quad (28)$$

Suppose that Ω is either a ball or an annulus. Again, suppose that f satisfy (a1)–(a3), and H satisfy (H1)–(H4). For each $\beta > 0$, denote by E_β and \mathcal{N}_β the associate energy functional and Nehari manifold. Take (u_β, v_β) to be a family of positive solutions of (28) minimizing $E_\beta|_{\mathcal{N}_\beta}$. Then, by the results shown in [7, 8], we know that there exists $\bar{u}, \bar{v} > 0$ such that $u_\beta \rightarrow \bar{u}, v_\beta \rightarrow \bar{v}$ strongly in $H_0^1(\Omega)$, and $\bar{w} := \bar{u} - \bar{v}$ satisfies

$$-\Delta \bar{w} = f(\bar{w}) \quad J(\bar{w}) = \min\{J(w) : w^\pm \neq 0, J'(w)w^+ = J'(w)w^- = 0\},$$

with $J(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 dx - \int_\Omega F(w) dx$. Thus \bar{w} is a *least energy nodal solution* which, by [1, Theorem 1.3], is known to be non radial. Therefore we conclude, from the strong convergence, that (u_β, v_β) are non radial solutions, at least for sufficiently large β .

5.1. An application within in a different variational setting. We close this paper with an application of Theorem 4.3 which does not fit in the framework of Theorem 1.3. Consider the cubic system

$$\begin{cases} -\Delta u = \lambda u - u^3 - \beta uv^2 & \text{in } \Omega \\ -\Delta v = \mu v - v^3 - \beta u^2 v & \text{in } \Omega \\ u, v \in H_0^1(\Omega), \quad u, v > 0 \text{ in } \Omega, \end{cases} \quad (29)$$

where we consider $\beta > 0$. Observe that due to the sign of the pure nonlinearities, this is not a particular case of (7). Following [6], in this case a minimal energy solutions is defined as a minimizer of the functional

$$I(u, v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx + \frac{1}{4} \int_\Omega (u^4 + v^4) dx + \frac{\beta}{2} \int_\Omega u^2 v^2 dx$$

constrained to the manifold

$$\mathcal{S} = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) : \int_\Omega u^2 dx = \int_\Omega v^2 dx = 1\}$$

(which represents a mass conservation law). With this framework, λ and μ are understood as Lagrange multipliers, and

$$\lambda = \lambda(u, v) = \int_\Omega (|\nabla u|^2 + u^4 + \beta u^2 v^2) dx, \quad \mu = \mu(u, v) = \int_\Omega (|\nabla v|^2 + v^4 + \beta u^2 v^2) dx. \quad (30)$$

By using direct methods and the maximum principle, it is easy to prove that (29) admits a positive solution, minimizer of $I|_{\mathcal{S}}$.

Theorem 5.5. *Let $u, v > 0$ be minimizers of $I|_{\mathcal{S}}$, hence in particular solutions of (29). Then u and v are foliated Schwarz symmetric with respect to antipodal points.*

Proof. We start with the observation that (u, v) solve (29) with λ, μ given by (30). For every $H \in \mathcal{H}_0$, by Lemma 4.2-(i) we deduce that $(u_H, v_{\hat{H}}) \in \mathcal{S}$. Moreover, Lemma 4.5 applied to the map $(u, v) \mapsto u^2 v^2$ gives

$$\begin{aligned} \min_{\mathcal{S}} I &\leq I(u_H, v_{\hat{H}}) = \frac{1}{2} \int_{\Omega} (|\nabla u_H|^2 + |\nabla v_{\hat{H}}|^2) dx + \frac{1}{4} \int_{\Omega} (u_H^4 + v_{\hat{H}}^4) dx + \frac{\beta}{2} \int_{\Omega} u_H^2 v_{\hat{H}}^2 dx \\ &= \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{1}{4} \int_{\Omega} (u^4 + v^4) dx + \frac{\beta}{2} \int_{\Omega} u^2 v^2 dx \\ &\leq \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx + \frac{1}{4} \int_{\Omega} (u^4 + v^4) dx + \frac{\beta}{2} \int_{\Omega} u^2 v^2 dx \\ &= I(u, v) = \min_{\mathcal{S}} I. \end{aligned} \tag{31}$$

Thus $(u_H, v_{\hat{H}}) \in \mathcal{S}$ and $I(u_H, v_{\hat{H}}) = \min_{\mathcal{S}} I$, and in particular $(u_H, v_{\hat{H}})$ solves (29) with

$$\lambda = \lambda(u_H, v_{\hat{H}}) = \int_{\Omega} (|\nabla u_H|^2 + u_H^4 + \beta u_H^2 v_{\hat{H}}^2) dx, \quad \mu = \mu(u_H, v_{\hat{H}}) = \int_{\Omega} (|\nabla v_{\hat{H}}|^2 + v_{\hat{H}}^4 + \beta u_H^2 v_{\hat{H}}^2) dx.$$

Again from (31) we deduce that actually

$$\int_{\Omega} u_H^2 v_{\hat{H}}^2 dx = \int_{\Omega} u^2 v^2 dx$$

and hence $\lambda(u, v) = \lambda(u_H, v_{\hat{H}})$, $\mu(u, v) = \mu(u_H, v_{\hat{H}})$. Thus (u, v) and $(u_H, v_{\hat{H}})$ solve the same system and Theorem 4.3 applies. \square

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